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Two and three loop sigma-model corrections to T-duality: Kasner and Schwarzschild

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Abstract: Two and three loop α' corrections are calculated for Kasner and Schwarzschild metrics, and their T-duals, in the bosonic string theory. These metrics are used to obtain the linear and quadratic α' corrections to T-duality. It is noted in particular that the inclusion of alpha' corrections and the requirement of consistency with the alpha'-corrected T-duality for the Kasner and Schwarzschild metrics enables one to fix uniquely the covariant form of the T-duality rules at three loops. As a generalization of the T-dual of the Schwarzschild geometry a class of massless geometries is presented.

Keywords: Bosonic Strings, String Duality, Black Holes in String Theory.

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Contents

1. Introduction and motivation

Target space duality was first introduced as a symmetry describing the interchange of the momentum and winding modes in the closed string compactified on a torus [1]. Later it was described as the symmetry of sigma models [2]. The linear α'-corrections to the T-duality rules are obtained in [3]–[5]. Further support for the α' expansion of T-duality is presented in [6] where the linear α' corrections to T-duality in the presence of torsion is obtained. Studying higher α' corrections to T-duality should provide a better understanding of both the mathematics of string theory in the curved space time and the pre-big bang scenario in string cosmology [7] where T-duality is an essential tool.

In this paper the three loop α' corrections to T-duality are computed in the critical bosonic string theory at the tree level of the string interaction for backgrounds composed of diagonal metric and dilaton. The paper is organized in the following way.

In the section 2 we are going to review the general diagonal Kasner backgrounds in $D = 26$. Since the Kasner background is of interest in Cosmology [8, 9] and particularly in Cosmological Billiard [10] calculating its string corrections should be interesting. We also present the β-function equations of the bosonic string theory.

In the section 3 we generalize the Kasner metric to a perturbative background in the critical bosonic string and we calculate the linear (two-loop) and the quadratic (three-loop) α' corrections to this background at the tree level of the string interaction.[1]

In the section 4 we write the Kasner metric on a periodic space-like directions and we apply T-duality in one direction to obtain the corresponding T-dual background. Next we add the linear and the quadratic α' corrections to the Kasner background and to its T-dual. We will observe that T-duality fails to relate the α'-corrected Kasner background to its α' corrected T-dual background. We will modify the rules by appropriate α' terms in such a way that the α' modified rules relate the α'-corrected Kasner background to its α' corrected T-dual background. Finally we will rewrite the α'-modifications in a Lorentz invariant form consistent with [4] to obtain the α' corrected T-duality rules for a general time-dependent background with diagonal metric and dilaton.

In the section 5 we review the Schwarzschild background in an arbitrary dimension. We introduce the time-dual of the Schwarzschild background by performing T-duality in the time direction of the related Euclidean geometry. We observe that the horizon of the Schwarzschild background changes into an intrinsic singularity under T-duality and the time-dual of the Schwarzschild metric is massless in $D = 4$. We introduce massless geometries in arbitrary dimensions for the low energy gravitational theory of bosonic string theory. We then calculate the linear and the quadratic α' corrections to both the Schwarzschild background and its T-dual background in $D = 4$ in the critical bosonic string theory at the tree-level of the string interaction. We observe that when the asymptotic behaviours of the fields are fixed at infinity the α' corrections generically diverge at the horizon of the Schwarzschild black hole.The possible relevance of this divergence is discussed.

Finally, by requiring that the α' modified T-duality relate the quadratic α' corrected Schwarzschild background to the quadratic α' corrected time-dual, we are able to identify uniquely the quadratic α' correction to the rule which describes the change of the dilaton under T-duality.

In the section 6 we provide a summary and possible generalizations of this work.

2. Kasner metrics and α' corrections

Exact conformal symmetry in string theory requires vanishing of the β functions of the corresponding sigma model. The vanishing of β-functions gives the equations of motion which describe the dynamics of the corresponding low energy theory. We will consider only the case of the critical bosonic string theory where the leading equations for the background composed of the metric and dilaton in the string frame read

$$0 = R_{ij} + 2 \ \nabla_i \nabla_j \ \phi \,, \tag{2.1}$$

$$0 = \Box\phi - (\nabla\phi)^2 + \frac{1}{4} \, R \,. \tag{2.2}$$

[1] The linear α' corrections to the Kasner metric for bosonic string theory were also studied in [11].

The stringy Kasner background [12] determined by the set of $\{p_1, \ldots, p_{25}\}$ as a solution of the above equations is

$$
\begin{aligned}
ds^2 &= -dt^2 + \sum_{i=1}^{25} t^{2p_i}\, dx_i^2\,, && (2.3)\\
\phi(t) &= \frac{\sum p - 1}{2}\, \ln t\,, \\
\sum_{i=1}^{25} p_i^2 &= 1\,, && (2.4)
\end{aligned}
$$

where satisfying the equations imposes the constraint (2.4). The intrinsic singularity at $t = 0$ of this metric shows up in the various scalar curvature terms

$$
\begin{aligned}
R &= \frac{(\sum p - 1)^2}{t^2}\,, && (2.5)\\
R_{\mu\nu}\, R^{\mu\nu} &= 2\, \frac{(\sum p - 1)^2}{t^4}\,, && (2.6)\\
R_{\mu\nu\lambda\eta}\, R^{\mu\nu\lambda\eta} &= \frac{6 + 2\, \sum p^4 - 8\, \sum p}{t^4}\,, && (2.7)\\
\nabla_\xi\, R_{\mu\nu\lambda\eta}\ \nabla^\xi\, R^{\mu\nu\lambda\eta} &= 16\, \frac{-\sum p^4 + 2 \sum p^3 + (\sum p^3)^2 - 2}{t^6}\,, && (2.8)
\end{aligned}
$$

where R, $R_{\mu\nu}$ and $R_{\mu\nu\lambda\eta}$ stand for the Ricci scalar, Ricci and Riemann tensors respectively.

The usual Kasner metric has $\phi = 0$ as $\sum p - 1 = 0$.[2] Here we allow for a more general configuration with a time dependent string coupling. The string coupling constant given by the local vacuum expectation value of the dilaton reads

$$
g_s = g_0\, e^{\,\phi} = g_0\, t^{(\sum p\, -\, 1)/2}\,. \qquad (2.9)
$$

Therefore for positive values of $\sum p - 1$ the string coupling constant vanishes at the time origin and diverges at infinity. For negative values of $\sum p - 1$ the string coupling constant diverges at the time origin and vanishes at infinity.

In this work we are interested in calculating the α' corrections at the tree level of the string interaction. The calculation at the tree level can be trusted as long as $g_s \ll 1$. For negative values of $\sum p - 1$ this condition is automatically satisfied at large t and for positive values of $\sum p - 1$ we set g_0 close to 0 to get $g_s \ll 1$ at the vicinity of a fixed large value of time where the perturbation in α' is going to be done.

In general one can calculate higher-loop corrections to the β-functions and obtain the α'-corrections to the equations of motion. Note that generally the β-functions are scheme dependent and the β-functions of various schemes should be mapped to each other by an appropriate field redefinition.

The β-functions of the cirital bosonic string theory for the backgrounds of dilaton and metric, calculated by the dimensional regularization method in the minimal subtraction

[2] The subleading β function (2.11) provides a non-zero value for the dilaton even if it vanishes at the leading order

scheme at three loops in α' read [13, 14],

$$
\begin{aligned}
\frac{1}{\alpha'}\beta_{ij} &= R_{ij} + 2\ \nabla_i \nabla_j\ \phi + \frac{1}{2}\ \alpha'\ R_{iklm} R_j{}^{\ klm} + \\
&\quad + \alpha'^2 \Big\{ \frac{1}{8}\ \nabla_k\ R_{ilmn}\ \nabla^k\ R_j{}^{\ lmn} - \frac{1}{16}\ \nabla_i\ R_{klmn}\ \nabla_j\ R^{klmn} + \frac{1}{2} R_{klmn}\ R_i{}^{\ mlp}\ R_j{}^{\ kn}{}_p - \\
&\qquad - \frac{3}{8}\ R_{iklj}\ R^{kmnp}\ R^l{}_{\ mnp} + \frac{1}{32}\ \nabla_j \nabla_i (\ R_{klmn}\ R^{klmn}) \Big\} = \ 0\,, \qquad (2.10) \\
\frac{1}{\alpha'}\beta_{\phi} &= -\frac{1}{2}\ \Box\ \phi + \partial_k \phi\ \partial^k \phi + \frac{1}{16}\ \alpha'\ R_{klmn}\ R^{klmn} + \\
&\quad + \alpha'^2 \Big\{ - \frac{3}{16}\ R^{kmnp}\ R^l{}_{\ mnp}\ \nabla_k \nabla_l \phi + \frac{1}{32}\ R_{klmn}\ R^{mnpq}\ R_{pq}{}^{\ kl} - \\
&\qquad - \frac{1}{24}\ R_{klmn}\ R^{qnpl}\ R_q{}^{\ m}{}_p{}^{\ k} + \frac{1}{64}\ \partial_i (R_{klmn}\ R^{klmn})\ \partial^i \phi \Big\} = 0\,, \qquad (2.11) \\
&\quad i, \ldots, q\ \in\ \{0, \ldots, 25\}\,.
\end{aligned}
$$

The α' terms in (2.10) and (2.11) are the string modifications to the metric and dilaton. In the following sections we refer to these equations to compute the α' corrections to the general Kasner background and the Schwarzschild metric in $D = 4$.

3. Perturbative α' corrections to the Kasner metric

In the previous section, the Kasner background was defined as the solution to the leading order β-function equations. Before moving to the subleading order corrections, one needs to generalize the Kasner background to a perturbative background in string theory. We implement this generalization by requiring that

1. String theory admits some time-dependent backgrounds where the metric is globally diagonal and the only non-vanishing field is the dilaton.

2. The above backgrounds admit a perturbative series expansion in α' i.e.

$$g_{\mu\nu}(t) \ = \ g^{(0)}_{\mu\nu}(t) + \alpha'\ g^{(1)}_{\mu\nu}(t) + \alpha'^2\ g^{(2)}_{\mu\nu}(t) + \cdots\,, \qquad (3.1)$$

$$\phi(t) \ = \ \phi^{(0)}(t) + \alpha'\ \phi^{(1)}(t) + \alpha'^2\ \phi^{(2)}(t) + \cdots\,, \qquad (3.2)$$

   where $g^{(0)}_{\mu\nu}(t)$ and $\phi^{(0)}(t)$ correspond to the Kasner background. All $g^{(n)}_{\mu\nu}$'s become automatically diagonal due to the first assumption.

We expect that for every given Kasner background there exists a string background satisfying the above conditions. In the next sections we are going to compute the linear and the quadratic α' corrections to the general Kasner background.

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3.1 The linear α' correction to the Kasner metric

We begin to investigate the α' corrections to the Kasner metric by making the following simple ansatz the generality of which we will verify at the end of this subsection.

$$\begin{aligned} ds^2 &= -dt^2 + \sum_{i=1}^{25} t^{2p_i} \left(1 + 2\frac{\alpha'}{t^2}\, b_i\right) dx_i^2 + O\left(\frac{\alpha'^2}{t^4}\right), && (3.3) \\ \phi(t) &= -\frac{1}{2}\left(1 - \sum_{i=1}^{25} p_i\right) \ln t + \frac{\alpha'\, B}{2\, t^2} + O\left(\frac{\alpha'^2}{t^4}\right), && (3.4) \\ \sum_{i=1}^{25} p_i^2 &= 1\,, \end{aligned}$$

where b_i's and B are some unknown constants numbers. Substituting (3.3) and (3.4) in (2.10) and keeping only the linear α' term results in the algebraic equations,

$$2\, b_i + \left(-\sum_{j=1}^{25} b_j + B\right) p_i + p_i^2 - p_i^3 = 0\,, \qquad (3.5)$$

$$6\left(-\sum_{i=1}^{25} b_i + B\right) - \sum_{i=1}^{25} p_i^4 - 1 + 2\sum_{i=1}^{25} p_i^3 + 4\sum_{i=1}^{25} p_i\, b_i = 0\,. \qquad (3.6)$$

Multiplying (3.5) by p_i and summing over i gives

$$2\sum_{i=1}^{25} b_i\, p_i + \left(-\sum_{i=1}^{25} b_i + B\right) + \sum_{i=1}^{25} p_i^3 - \sum_{i=1}^{25} p_i^4 = 0\,. \qquad (3.7)$$

(3.7) and (3.6) are solved by

$$B - \sum_{i=1}^{25} b_i = \frac{1}{4}\left(1 - \sum_{i=1}^{25} p^4\right). \qquad (3.8)$$

Using (3.5) and (3.8) one easily obtains

$$\begin{aligned} b_i &= -p_i\left(\frac{1 - \sum_{j=1}^{25} p_j^4}{8} + \frac{1}{2}(p_i - p_i^2)\right), && (3.9) \\ B &= \left(1 - \sum_{j=1}^{25} p_j^4\right)\left(\frac{1}{4} - \frac{\sum_{i=1}^{25} p_i}{8}\right) - \frac{1}{2}\left(1 - \sum_{i=1}^{25} p_i^3\right). && (3.10) \end{aligned}$$

The results obtained for b_i and B satisfies (2.11) as well. Now let us investigate the general solution by writing the corrections in the following form

$$\begin{aligned} ds^2 &= -dt^2 + \sum_{i=1}^{25} t^{2p_i} \left(1 + \frac{2\alpha'}{t^2}\,(b_i(t) + b_i)\right) dx_i^2 + O\left(\frac{\alpha'^2}{t^4}\right), && (3.11) \\ \phi(t) &= \frac{1}{2}\left(\sum p - 1\right) \ln t + (B + B(t))\,\frac{\alpha'}{2\, t^2} + O\left(\frac{\alpha'^2}{t^4}\right), && (3.12) \end{aligned}$$

where b_i and B are given respectively in (3.9) and (3.10). As we have already solved the inhomogeneous equations at order α' to determine the role of possible $b_i(t)$ and $B(t)$ it is sufficient to consider (3.11) and (3.12) in the tree-level equations (2.1) and (2.2). This gives the general solution for $b_i(t)$ and $B(t)$:

$$b_i(t) = -\, p_i\, c_1\, t + c_i^{(1)}\, t^2 + 2\, t^2\, c_i^{(2)}\, \ln t\,, \tag{3.13}$$

$$B(t) = c_1 \left(1 - \sum p\right)\, t + c_2\, t^2 + 2\, t^2\, \ln t\, \sum c_i^{(2)}\,, \tag{3.14}$$

$$\sum c_i^{(2)}\, p_i = 0\,, \tag{3.15}$$

where c_1, c_2, c_i^1's and c_i^2's are constants of integration. At the first sight the appearance of these constants of integration may seem disappointing, however a closer look shows that

- c_1 corresponds to an infinitesimal time displacement , $t \;\rightarrow\; t - \alpha'\, c_1$.
- c_2 corresponds to a constant shift in the dilaton field.
- $c_i^{(1)}$'s correspond to proper scaling in the x_i directions..
- $c_i^{(2)}$ describes an infinitesimal change in p_i, $p_i \rightarrow p_i + 2\alpha' c_i^{(2)}$, constrained to $\sum p_i^2 = 1$ (3.15).

Therefore all the arbitrary constants in (3.13) and (3.14) are either infinitesimal redefinitions of variables which govern the leading asymptotic behavior of the fields at infinity or correspond to a shift in time or dilaton. We fix the leading asymptotic behavior of the metric and dilaton to their corresponding terms at tree-level and we set all of the above constants to zero. Doing so we obtain (3.9) and (3.10) as the values of the linear α' corrections to the metric and the dilaton.[3]

3.2 The quadratic α' correction to the Kasner metric

Similar to what was done in the previous section the quadratic α' correction to the metric and the dilaton may be written as

$$ds^2 = -dt^2 + \sum_{i=1}^{25} t^{2p_i} \left(1 + 2\, \frac{\alpha'}{t^2}\, b_i + 2\, \frac{\alpha'^2}{t^4}\, c_i\right) + O\left(\frac{\alpha'^3}{t^6}\right), \tag{3.16}$$

$$\phi(t) = \frac{\sum p - 1}{2}\, \ln t + \frac{\alpha'}{2\, t^2}\, B + \frac{\alpha'^2}{t^4}\, W + O\left(\frac{\alpha'^3}{t^6}\right), \tag{3.17}$$

where c_i's and W are some unknown constant numbers and b_i's and B are written explicitly in the (3.9) and (3.10). The equation that comes from the time-time component of (2.10)

[3] In the Heterotic string theory the β functions at two-loop in α' are the same as the ones of bosonic string theory by replacing α' with $\frac{\alpha'}{2}$. Therefore the results obtained in (3.9) and (3.10) trivially can be extended to Heterotic string theory.

reads

$$0 = -\frac{17}{4}\sum p^3 \sum p^4 - \sum p^5 \sum p^4 - \frac{\sum p^3 \, (\sum p^4)^2}{8} + \\ + \sum \left(-\frac{45}{8}\, p^3 + \frac{17}{4}\, p^4 - 19\, p^5 + 11\, p^6 - 2\, p^7 + 14\, p^4 \right) + \\ + \frac{11}{4} + 8 \sum c\, p + 20 \left(2\, W - \sum c \right) . \tag{3.18}$$

The remaining equations generated by (2.10) are

$$0 = \frac{3}{16} p_i - \frac{1}{4}\, p_i^2 + 7\, p_i^3 - 18\, p_i^4 + 16\, p_i^5 - 4\, p_i^6 + \\ + p_i \sum \left(p^6 - 2\, p^5 + \frac{5}{2}\, p^3 - \frac{15}{8}\, p^4 \right) + \frac{11}{16}\, p_i \left(\sum p^4 \right)^2 - \\ - p_i^2 \left(\frac{7}{2} \sum p^4 + \frac{1}{4} \left(\sum p^4 \right)^2 \right) + 5\, p_i^3 \sum p^4 - 2\, p_i^4 \sum p^4 - \\ - \frac{p_i}{2} \sum p^3 \sum p^5 + 4 \left(2\, W - \sum c \right) p_i + 16\, c_i \; . \tag{3.19}$$

Multiplying (3.19) with p_i and summing over i gives

$$0 = -4 \sum p^3 \sum p^4 - 2 \sum p^5 \sum p^4 - \frac{1}{4} \sum p^3 \left(\sum p^4 \right)^2 + \\ + \sum \left(-20\, p^5 + 17\, p^6 + \frac{9}{4}\, p^3 + \frac{41}{8}\, p^4 - 4\, p^7 \right) + \\ + \frac{3}{16} + \frac{91}{16} \left(\sum p^4 \right)^2 + 16 \sum c\, p + 4 \left(2\, W - \sum c \right) . \tag{3.20}$$

From (3.19) and (3.20) one obtains

$$2\, W - \sum c = \sum \left(\frac{3}{8}\, p^3 - \frac{61}{96}\, p^4 + \frac{1}{2}\, p^5 - \frac{5}{36}\, p^6 \right) + \\ + \frac{1}{8} \sum p^4 \sum p^3 - \frac{5}{64} \left(\sum p^4 \right)^2 - \frac{85}{576} \, . \tag{3.21}$$

Substituting (3.21) in (3.19) identifies the c_i giving

$$c_i = p_i \left\{ \frac{1}{4}\, p_i^5 - p_i^4 + \frac{9}{8}\, p_i^3 - \frac{7}{16}\, p_i^2 + \frac{1}{64}\, p_i + \frac{29}{1152} + \right. \\ + p_i \left(\frac{7}{32} + \frac{1}{8}\, p_i^2 - \frac{5}{16}\, p_i \right) \sum p^4 + \frac{1}{64}\, p_i \left(\sum p^4 \right)^2 + \\ \left. + \sum \left(\frac{53}{192}\, p^4 - \frac{1}{36}\, p^6 \right) - \frac{3}{128} \left(\sum p^4 \right)^2 \underbrace{- \frac{1}{4} \sum p^3}_{} \right\} , \tag{3.22}$$

$$W = -\frac{\sum p^3}{32} + \frac{17}{48} \sum p^4 - \frac{\sum p^5}{4} + \frac{\sum p^6}{18} + \frac{1}{32} \left(\sum p^4 \right)^2 - \frac{3}{32} \sum p^3 \sum p^4 + \\ + \left\{ \frac{53 \sum p^4}{384} - \frac{\sum p^3}{8} - \frac{3 (\sum p^4)^2}{256} + \frac{29}{2304} - \frac{\sum p^6}{72} \right\} \sum p - \frac{19}{288} \, . \tag{3.23}$$

This gives us the general α'^2 corrections up to possible redefinition of variables as discussed in the previous subsection.

4. T-Duality and the α' corrections

In this section we are going to obtain the quadratic α' modifications to the T-duality rules for a time-dependent background composed of a diagonal metric and dilaton, consistent with the results of the previous sections on the α' corrections to the Kasner metric and its T-dual.

We study T-duality in the effective field theory where it is a map from on-shell field contents of a given space to the on-shell field contents of its corresponding T-dual space. In order to represent this map explicitly, we first represent both the field contents of a space "$ds^2, \phi(t)$" and the field contents of the corresponding T-dual space "$d\tilde{s}^2, \tilde{\phi}(t)$" in the co-moving frame i.e.

$$\begin{cases} ds^2 \,=\, -\,dt^2 \,+\, ds_\perp^2\,, \\ \quad \phi \,=\, \phi(t)\,, \end{cases} \qquad \begin{cases} d\tilde{s}^2 \,=\, -\,dt^2 \,+\, d\tilde{s}_\perp^2\,, \\ \quad \tilde{\phi} \,=\, \tilde{\phi}(t)\,. \end{cases} \tag{4.1}$$

It should be noticed that the definition of the metric and the dilaton are always fixed in such a way that the corresponding β-functions are provided by (2.10) and (2.11).

In the following we write the Kasner metric on a periodic space-like direction and we apply T-duality in x_{25} direction to obtain the corresponding T-dual background. Next we add the α' corrections to the Kasner background and to its T-dual. We will observe that T-duality fails to relate the α'-corrected Kasner background to its α' corrected T-dual background. We will modify the rules by appropriate α' terms in such a way that the α' modified rules relate the α'-corrected Kasner background to its T-dual. At the end of this section we will rewrite the α'-modification in a Lorentz invariant form consistent with [4] to obtain the α' corrected T-duality rules for a general time-dependent background with diagonal metric and dilaton.

Let us start the calculation by writing the Kasner background on periodic space directions

$$ds^2 \,=\, -\,dt^2 + \sum_{i=1}^{25} t^{2p_i}\,\left(\frac{r_i}{\sqrt{\alpha'}}\right)^2\, dx_i^2\,, \tag{4.2}$$

$$\phi(t) \,=\, \frac{\sum p - 1}{2}\,\ln t\,, \tag{4.3}$$

$$\sum_{i=1}^{25} p_i^2 \,=\, 1\,, \tag{4.4}$$

$$x_i \,\equiv\, x_i + 2\,\pi\,, \tag{4.5}$$

where each x_i is compactified on a circle with time dependent radius $r_i(t) = r_i\; t^{p_i}$.

For the Kasner background the rules that describe T-duality in x_{25} read [4]

$$\ln \tilde{g}_{25\;25} \,=\, -\,\ln g_{25\;25}\,, \tag{4.6}$$

$$\ln \tilde{g}_{ii} \,=\, \ln g_{ii} \qquad i \in \{1, \ldots, 24\}\,, \tag{4.7}$$

$$\tilde{\phi} - \frac{1}{4}\,\ln\det\tilde{g} \,=\, \phi - \frac{1}{4}\,\ln\det g\,. \tag{4.8}$$

[4]These rules are written in such a way that it is manifest that $T^2 = 1$, where T represents the T-duality

Applying the above rules on the Kasner background returns its T-dual background,

$$d\tilde{s}^2 = -\,dt^2 + \sum_{i=1}^{24} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 d\tilde{x}_i^2 + t^{-2p_{25}} \left(\frac{r_{25}}{\sqrt{\alpha'}}\right)^{-2} d\tilde{x}_{25}^2\,, \tag{4.9}$$

$$\tilde{\phi}(t) = -\,\ln\left(\frac{r_{25}}{\sqrt{\alpha'}}\right) + \left(\sum_{i=1}^{24} p_i \,-\, p_{25} - 1\right) \frac{\ln t}{2}\,, \tag{4.10}$$

$$\tilde{x}_i \equiv \tilde{x}_i + 2\,\pi\,.$$

The dual background is still a Kasner background where x_{25} is compactified on a circle with radius $\tilde{r}_{25}(t) = \frac{\alpha'}{r_{25}}\; t^{-p_{25}}$ [15]. Now let the α' corrections be added to the Kasner background

$$ds^2 = -dt^2 + \sum_{i=1}^{25} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 \left\{1 + \frac{2\alpha'}{t^2}\, b_i + \frac{2\alpha'^2}{t^4} c_i\right\} dx_i^2 + O\left(\frac{\alpha'^3}{t^6}\right) \tag{4.11}$$

$$\phi(t) = \left(\sum_{i=1}^{24} p_i \,+\, p_{25} \,-\, 1\right) \frac{\ln t}{2} + \frac{\alpha'}{2\,t^2} B + \frac{\alpha'^2}{t^4}\, W \,+\, O\left(\frac{\alpha'^3}{t^6}\right), \tag{4.12}$$

where b_i, B, c_i and W respectively are identified by (3.9), (3.10), (3.22) and (3.23) for the set of $(p_1, \ldots, p_{24}, p_{25})$. Adding the corresponding α' corrections to the dual background gives

$$\begin{aligned} d\tilde{s}^2 &= -dt^2 + \sum_{i=1}^{24} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 \left\{1 + \frac{2\alpha'}{t^2}\, \tilde{b}_i + \frac{2\alpha'^2}{t^4} \tilde{c}_i\right\} dx_i^2 + \\ &\quad + t^{-2p_{25}} \left(\frac{r_{25}}{\sqrt{\alpha'}}\right)^{-2} \left\{1 + \frac{2\alpha'}{t^2}\, \tilde{b}_{25} + \frac{2\alpha'^2}{t^4} \tilde{c}_{25}\right\} dx_{25}^2\,, \end{aligned} \tag{4.13}$$

$$\tilde{\phi}(t) = -\,\ln\left(\frac{r_{25}}{\sqrt{\alpha'}}\right) + \left(\sum_{i=1}^{24} p_i \,-\, p_{25} \,-\, 1\right) \frac{\ln t}{2} + \frac{\alpha'}{2\,t^2} \tilde{B} + \frac{\alpha'^2}{t^4}\, \tilde{W} \,+\, O\left(\frac{\alpha'^3}{t^6}\right), \tag{4.14}$$

where $\tilde{b}_i$, $\tilde{B}$, $\tilde{c}_i$ and $\tilde{W}$ respectively are provided by (3.9), (3.10), (3.22) and (3.23) for the set of $(p_1, \ldots, p_{24}, -p_{25})$. It should be noticed that for $1 \le i \le 24$ we have $b_i = \tilde{b}_i$ but due to the presence of the under-braced term in (3.22) $c_i \neq \tilde{c}_i$.

If we insert (4.13, 4.11) in the T-duality rules (4.6, 4.7, 4.8) we obtain[5]

$$\ln \tilde{g}_{25\;25} + \ln g_{25\;25} = -\,\frac{2\;\alpha'}{t^2}\; p_{25}^2 + O\left(\frac{\alpha'^2}{t^4}\right)\,, \tag{4.15}$$

$$\ln \tilde{g}_{ii} - \ln g_{ii} = \frac{\alpha'^2}{t^4}\; p_i\; p_{25}^3 + O\left(\frac{\alpha'^3}{t^6}\right)\,, \tag{4.16}$$

$$\tilde{\phi} - \phi + \frac{1}{4}\; \ln \frac{\det g}{\det \tilde{g}} = -\frac{\alpha'^2}{2\;t^4}\; p_{25}^3 + O\left(\frac{\alpha'^3}{t^6}\right), \tag{4.17}$$

These relations indicate that none of the T-duality rules are satisfied and they all must

[5]Only the linear α' term in (4.15) is kept. After finding the linear α' term in (4.24) the quadratic term is fixed in (6.7) .

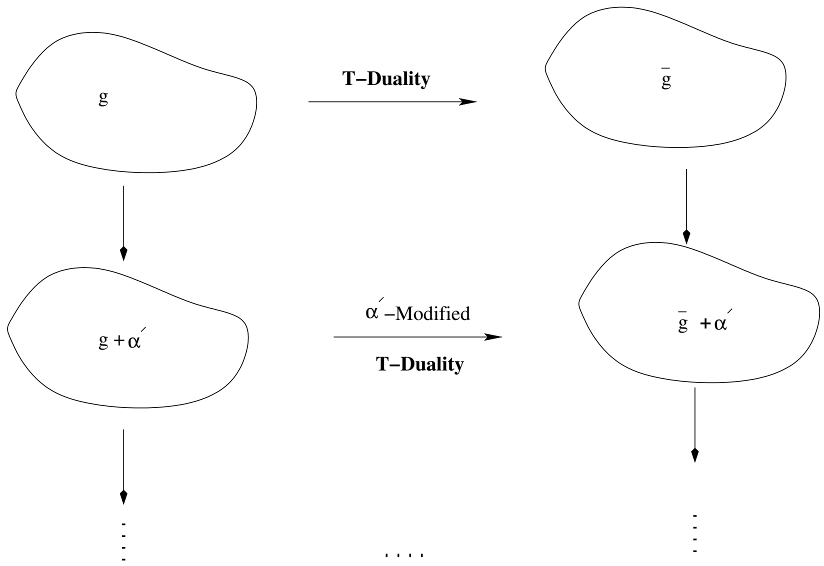

Figure 1: T-duality and the α' corrections. By requiring that α' modified T-duality maps the α' corrected dual backgrounds to each other the α' modifications to the rules of the T-duality can be identified.

be modified by appropriate α' terms (Fig.1). In order to find the simplest modifications let (4.15), (4.16) and (4.17) be written in the following forms

$$\ln \tilde{g}_{25\,25} + \frac{\alpha'}{t^2}\tilde{p}_{25}^2 = -\left(\ln g_{25\,25} + \frac{\alpha'}{t^2}p_{25}^2\right) + O\left(\frac{\alpha'^2}{t^4}\right), \tag{4.18}$$

$$\tilde{g}_{ii} + \frac{\alpha'^2}{2\,t^4}\,\tilde{p}_i\,\tilde{p}_{25}^3 = \ln g_{ii} + \frac{\alpha'^2}{2\,t^4}\,p_i\,p_{25}^3 + O\left(\frac{\alpha'^3}{t^6}\right), \tag{4.19}$$

$$\tilde{\phi} - \frac{1}{4}\,\ln\det\tilde{g} - \frac{\alpha'^2}{4\,t^4}\,\tilde{p}_{25}^3 = \phi - \frac{1}{4}\,\ln\det g - \frac{\alpha'^2}{4\,t^4}\,p_{25}^3 + O\left(\frac{\alpha'^3}{t^6}\right), \tag{4.20}$$

where $\tilde{p}_i = p_i$ and $\tilde{p}_{25} = -p_{25}$. The simplest scheme to define the α' corrected T-duality is to accept the above relations as the rules of the α'-modified T-duality. We follow this strategy and we accept (4.18), (4.19) and (4.20) as the rules of the leading α' modified T-duality.

The leading α'-modified T-duality explicitly depends on the p_i's. These terms can be written as polynomials in derivatives of the metric and dilaton, in the following covariant forms[6]

$$\frac{p_i\,p_{25}^3}{t^4} = \frac{1}{16}\,\nabla_\mu \ln g_{ii}.\nabla^\mu \ln g_{25\,25}\,\nabla_\nu \ln g_{25\,25}.\nabla^\nu \ln g_{25\,25} + O(\alpha'), \tag{4.21}$$

$$\frac{p_{25}^2}{t^2} = -\frac{1}{4}\nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25} + O(\alpha'), \tag{4.22}$$

$$-\frac{p_{25}^3}{4\,t^4} = \frac{A}{32}\,\nabla_\mu \ln g_{25\,25}\,\nabla_\nu \ln g_{25\,25}\,\nabla^\mu\nabla^\nu \ln g_{25\,25} +$$

[6]In this notation, the covariant derivative acts on the logarithm of the metric as if it were a scalar.

$$+\ \frac{B}{16}\ \nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}.\nabla^\nu \left(\phi - \frac{1}{4}\ln\det g\right) +$$
$$+\ \frac{C}{32}\ \nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25}\ \Box \ln g_{25\,25} + O(\alpha')\,, \tag{4.23}$$

where A, B and C are real numbers satisfying $A+B+C=1$. Using the above identities gives the leading α' modified T-duality rules on the metric

$$\ln\tilde{g}_{25\,25} - \frac{\alpha'}{4}\tilde{\nabla}_\mu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln\tilde{g}_{25\,25} =$$
$$= -\left\{\ln g_{25\,25} - \frac{\alpha'}{4}\nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25}\right\} + O(\alpha'^2\nabla^4)\,, \tag{4.24}$$
$$\ln\tilde{g}_{ii} + \frac{\alpha'^2}{32}\tilde{\nabla}_\mu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln\tilde{g}_{ii}\ \tilde{\nabla}_\nu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\nu \ln\tilde{g}_{25\,25} =$$
$$= \ln g_{ii} + \frac{\alpha'^2}{32}\nabla_\mu \ln g_{ii}.\nabla^\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}.\nabla^\nu \ln g_{25\,25} + O(\alpha'^3\nabla^6)\,. \tag{4.25}$$

The T-duality rule which describes the change in the dilaton reads

$$\tilde{\phi} - \frac{1}{4}\ln\det\tilde{g} + \frac{\alpha'^2 A}{32}\ \tilde{\nabla}_\mu \ln\tilde{g}_{25\,25}\ \tilde{\nabla}_\nu \ln\tilde{g}_{25\,25}\ \tilde{\nabla}^\mu\tilde{\nabla}^\nu \ln\tilde{g}_{25\,25} +$$
$$+\frac{\alpha'^2 B}{16}\ \tilde{\nabla}_\mu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln\tilde{g}_{25\,25}\ \tilde{\nabla}_\nu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\nu \left(\tilde{\phi} - \frac{1}{4}\ln\det\tilde{g}\right) +$$
$$+\frac{\alpha'^2 C}{32}\ \tilde{\nabla}_\mu \ln\tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln\tilde{g}_{25\,25}\ \tilde{\Box} \ln\tilde{g}_{25\,25} =$$
$$=\ \phi - \frac{1}{4}\ln\det g + \frac{\alpha'^2 A}{32}\ \nabla_\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}\ \nabla^\mu\nabla^\nu \ln g_{25\,25} +$$
$$+\ \frac{\alpha'^2 B}{16}\ \nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}.\nabla^\nu \left(\phi - \frac{1}{4}\ln\det g\right) +$$
$$+\ \frac{\alpha'^2 C}{32}\ \nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25}\ \Box \ln g_{25\,25} + O(\alpha'^3\nabla^6)\,, \tag{4.26}$$

where $A+B+C=1$.

The above rules (4.24), (4.25) and (4.26) are written in Lorentz invariant forms and they describe T-duality on backgrounds composed of diagonal metric and dilaton given that the fields are in the string frame and the co-moving frame . These rules are in agreement with those of [3] and [4] where only the linear α' modifications were considered.

One observes that redefining the metrics g_{ii} and $\tilde{g}_{ii}$ to g^*_{ii} and $\tilde{g}^*_{ii}$ in the following way

$$g^*_{ii} = g_{ii}\ \exp\left(\frac{\alpha'^2}{32}\nabla_\mu \ln g_{ii}.\sum_{k=1}^{25}\nabla^\mu \ln g_{kk}\ \nabla_\nu \ln g_{kk}.\nabla^\nu \ln g_{kk}\right), \tag{4.27}$$
$$\tilde{g}^*_{ii} = \tilde{g}_{ii}\ \exp\left(\frac{\alpha'^2}{32}\tilde{\nabla}_\mu \ln\tilde{g}_{ii}.\sum_{k=1}^{25}\tilde{\nabla}^\mu \ln\tilde{g}_{kk}\ \tilde{\nabla}_\nu \ln\tilde{g}_{kk}.\tilde{\nabla}^\nu \ln\tilde{g}_{kk}\right), \tag{4.28}$$

compensates the α' corrections to the rules describing the change of the metric in the transverse directions under T-duality in any direction. For the Kasner metric this redefinition corresponds to choosing different coordinates on the space and its T-dual space. In general this transformation implies that one really needs a field redefinition to rewrite the higher order T-duality rules in the same form as the leading order T-duality rules.

Jack and Parson in [16] have calculated the same corrections to T-duality and proved that $O(d,d)$ invariance of the conformal invariance condition, observed at one loop [17] and two loops [18], can be preserved also at three loops given an appropriate field redefinition and coordinate transformation (either on the background or its T-dual background but not both). They did not explicitly provide the modification to T-duality but rewriting their results for the case of the Kasner background reproduces (4.25).

In this work *we do not redefine the metric and the dilaton* and we maintain the convention given by the dimensional regularization method in the minimal subtraction scheme which is the same as fixing the definition of the metric and dilaton in such a way that the corresponding β functions are provided by (2.10) and (2.11). This convention implies (4.25) which means that applying T-duality in one direction alters the metric in all directions.

Sometimes writing the fields explicitly in the co-moving frame is not easy. Thus we are going to write the α' T-duality rules in an alternative frame. As a simple generalization of the co-moving frame let us introduce an "*almost co-moving frame*" with a "*characteristic function (f)*" in which the metrics read

$$ds^2 \;=\; -\,f(T)\,dT^2 \,+\, ds_\perp^2\,, \qquad \phi \;=\; \phi(T) \tag{4.29}$$

$$d\tilde{s}^2 \;=\; -\,f(T)\,dT^2 \,+\, d\tilde{s}_\perp^2\,, \qquad \tilde{\phi} \;=\; \tilde{\phi}(T)\,, \tag{4.30}$$

where $f(T)$ can be an arbitrary function of time. The α' modified T-duality rules in the almost co-moving frame for a general characteristic function are provided by those of the co-moving frame if within (4.26) we replace $\det g$ and $\det \tilde{g}$ respectively by $\det^* g$ and $\det^* \tilde{g}$ given below

$$\det{}^* g = \frac{\det g}{f(T)}\,, \tag{4.31}$$

$$\det{}^* \tilde{g} = \frac{\det \tilde{g}}{f(T)}\,. \tag{4.32}$$

For a specific background, one may choose an appropriate characteristic function to simplify the computations.

5. The linear and the quadratic α' corrections to the Schwarzschild background and its T-dual in $D = 4$

In this section the Schwarzschild background in an arbitrary dimension is reviewed. The time-dual of Schwarzschild background is introduced by performing a non-compact T-duality in the time direction of the Schwarzschild metric in the region outside the black hole horizon. We must note the this non-compact T-duality in a time-like direction is not on the same footing as the usual T-duality, but it has been studied in [19]. The linear and the quadratic α' corrections to the Schwarzschild background and its time-dual in $D = 4$ are computed. Requiring (4.26) to relate the quadratic α' corrected Schwarzschild background to the quadratic α' corrected its time-dual we are able to identify the unknown coefficients in (4.26) with the values $A = 1$ and $B = C = 0$.

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The Schwarzschild background in D dimensions, $(D > 3)$, is given by

$$\begin{aligned} ds^2 &= -\left(1 - \frac{1}{r^{D-3}}\right) dt^2 + \frac{dr^2}{1 - \frac{1}{r^{D-3}}} + r^2\, d\Omega_{D-2}\,, \\ \phi(r) &= 0\,, \end{aligned} \tag{5.1}$$

where the mass has been chosen to give simply a coefficient of one in the metric and we will maintain this convention in the following sections.

The intrinsic singularity of the Schwarzschild metric shows itself in the following scalar curvature terms

$$R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta} = \frac{(D-1)\,(D-2)^2\,(D-3)}{r^{2D-2}}\,. \tag{5.2}$$

Applying T-duality to the time direction, on the metric outside the horizon, we get the T-dual background (denoted from hereon by a tilde) is,

$$\begin{aligned} d\tilde{s}^2 &= -\frac{dt^2}{1 - \frac{1}{r^{D-3}}} + \frac{dr^2}{1 - \frac{1}{r^{D-3}}} + r^2\, d\Omega_{D-2}\,, \\ \tilde{\phi}(r) &= -\frac{1}{2}\ln\left(1 - \frac{1}{r^{D-3}}\right)\,. \end{aligned} \tag{5.3}$$

The above background solves (2.1) and (2.2). We refer to this background as the " time-dual of the Schwarzschild background". The singularities of the time-dual of the Schwarzschild metric can be seen in its Ricci scalar curvature,

$$\tilde{R} = -\frac{(D-3)^2}{r^{D-1}\,(r^{D-3} - 1)}\,. \tag{5.4}$$

Therefore the time-dual of the Schwarzschild metric has two intrinsic singularities at $r = 0, 1$ which means in particular that the coordinate singularity of the Schwarzschild metric at the horizon has changed to an intrinsic singularity in its time-dual metric. This behaviour is similar to what have been observed in [3] and in [20].

In order to get a better understanding of the time-dual of the Schwarzschild background let us write it in the Einstein frame[7],

$$\begin{aligned} d\tilde{s}^2_E &= \left(1 - \frac{1}{r^{D-3}}\right)^{\frac{4-D}{D-2}} (-\,dt^2 + dr^2) + r^2\left(1 - \frac{1}{r^{D-3}}\right)^{\frac{2}{D-2}} d\Omega_{D-2}\,, \\ \tilde{\phi}(r) &= -\frac{1}{2}\ln\left(1 - \frac{1}{r^{D-3}}\right)\,. \end{aligned} \tag{5.5}$$

The singularities of the time-dual of the Schwarzschild metric in the Einstein frame are the same as the ones of the string frame because the Ricci scalar of (5.5) reads

$$\tilde{R}_E = \frac{(D-3)^2}{D-2}\,\frac{1}{r^{2D-4}}\left(\frac{r^{D-3}}{r^{D-3} - 1}\right)^{\frac{D}{D-2}}\,. \tag{5.6}$$

[7] The Einstein frame is obtained by $g_{\mu\nu} \rightarrow g_{\mu\nu}\, e^{-\frac{4\phi}{D-2}}$

The time-dual of the Schwarzschild metric in the Einstein frame for $D > 4$ describes a geometry with two singularities and a newtonian mass proportional to $(4-D)/(D-2)$. In $D = 4$ the time-dual of the Schwarzschild metric is massless and in the Einstein frame reads

$$\begin{aligned} ds^2 &= -\,dt^2 + dr^2 + r\,|r-1|\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \\ \phi(r) &= -\frac{1}{2}\ln\left|1-\frac{1}{r}\right|\,. \end{aligned} \tag{5.7}$$

The above metric (5.7), describes a geometry with a naked singularity at $r = 1$ and a vanishing newtonian mass. In supergravity similar geometries are studied and named massless black(white) holes [21]–[24]. Here (5.7) represents the corresponding object in the low energy gravitational theory of the bosonic string theory compactified on T^{22} to $D = 4$. Other similar objects in higher dimensions within the low energy theory of the bosonic string theory can be found. For example the background provided below

$$ds^2 = -\,dt^2 + \frac{dr^2}{1+\left(\frac{q}{r^{D-3}}\right)^2} + r^2\,d\Omega_{D-2} + dx^2_{26-D} \tag{5.8}$$

$$\phi = \pm\frac{D-2}{2\sqrt{D-3}}\,\mathrm{ArcSinh}\left(\frac{q}{r^{D-3}}\right) + \phi_0\,, \tag{5.9}$$

solves the leading order equations of motion in the Einstein frame.

$$R_{\mu\nu} - \frac{4}{D-2}\,\nabla_\mu\phi\nabla_\nu\phi = 0\,, \tag{5.10}$$

$$\Box\phi = 0\,. \tag{5.11}$$

This background (5.8), represents a geometry with a vanishing newtonian mass and an intrinsic singularity at "$r = 0$" as is clear from the corresponding Ricci scalar curvature term

$$R = (D-2)\,(D-3)\,\left(\frac{q}{r^{D-3}}\right)^2\,, \tag{5.12}$$

We see that q parameterizes the fall off of the fields at infinity and to ensure that the dilaton is real we must choose q to also be real. We then see that there is a naked singularity at $r = 0$.

5.1 The linear and the quadratic α' corrections to the Schwarzschild background in $D = 4$

The Schwarzschild metric in $D = 3 + 1$ reads

$$ds^2 = -(1-\frac{1}{r})\;dt^2 + \frac{1}{1-\frac{1}{r}}\;dr^2 + r^2\;(d\theta^2 + \sin^2\theta\;d\phi^2)\,, \tag{5.13}$$

where its intrinsic singularity can be seen in the Ricci scalar curvature

$$R_{\mu\nu\lambda\eta}\,R^{\mu\nu\lambda\eta} = \frac{12}{r^6}\,. \tag{5.14}$$

The Schwarzschild metric can be generalized to a perturbative consistent background of the bosonic string theory in the critical dimension by assuming that

1. The critical bosonic string theory admits the following background

$$ds^2 = -\, g_{tt}(r)\, dt^2 \,+\, g_{rr}(r)\, dr^2 \,+\, g_\Omega(r)\,(d\theta^2 + \sin^2\theta\; d\phi^2) + dx_\perp^2\,, \tag{5.15}$$
$$\phi = \phi(r)\,, \tag{5.16}$$

where $dx_\perp^2$ and ϕ respectively represent the 22-dimensional flat space and the dilaton.

2. Within the above background the metric and the dilaton admit the following perturbative series in α'

$$g_{tt}(r) = \left(1 - \frac{1}{r}\right)\;\left(1 + \alpha'\; g_{tt}^{(1)}(r) +\; \alpha'^2\; g_{tt}^{(2)}(r) + \cdots\right), \tag{5.17}$$
$$g_{rr}(r) = \frac{1}{1-\frac{1}{r}}\;\left(1 + \alpha'\; g_{rr}^{(1)}(r) + \alpha'^2\; g_{rr}^{(2)}(r) + \cdots\right), \tag{5.18}$$
$$g_\Omega(r) = r^2\,, \tag{5.19}$$
$$\phi(r) = \;0 + \alpha'\; \phi^{(1)}(r) + \alpha'^2\; \phi^{(2)}(r) + \cdots\,. \tag{5.20}$$

Using the β-functions (2.10),(2.11) gives the general solution for the linear α' corrections

$$\phi^{(1)}(r) = -\frac{2\,+\,3\,r\,+\,6\,r^2}{12\,r^3} \;-\; \left(c_3 \,+\, \frac{1}{2}\right)\,\ln\left(1-\frac{1}{r}\right) \,+\, c_1\,, \tag{5.21}$$
$$g_{rr}^{(1)}(r) = \frac{10\,-\,3\,r\,-\,6\,r^2\,+\,12\,(c_2+4\,c_3)\,r^3}{12\,r^3\,(r\,-\,1)} \;-\; (c_3+\frac{1}{2})\frac{\ln(1-\frac{1}{r})}{r-1}\,, \tag{5.22}$$
$$g_{tt}^{(1)}(r) = \frac{6\,+\,5\,r\,+\,12\,r^2\,-\,12\,r^3\,-\,12\,(c_2\,+\,2\,c_3)\,r^3\,+\,c_4\,(r^4\,-\,r^3)}{12\,r^3\,(r\,-\,1)} \;+$$
$$+\,(3\,-\,2\,r)\,\left(c_3\,+\,\frac{1}{2}\right)\;\frac{\ln(1\,-\,\frac{1}{r})}{r\,-\,1}\,.$$

In [25] and many following works a particular boundary conditions were chosen for the metric and dilaton. In these works it was assumed that after choosing (5.17) and (5.18) as the perturbative series for the metric then the α' corrections to the metric are finite at the horizon of the black hole and the dilaton vanishes at infinity, [8]

$$\begin{aligned} \phi^{\star(1)}(r)|_{r=\infty} &= 0\,,\\ g_{tt}^{\star(1)}(r)|_{r=1} &< \infty\,,\\ g_{rr}^{\star(1)}(r)|_{r=1} &< \infty\,. \end{aligned} \tag{5.23}$$

The above boundary conditions fix the constants of the integration to values of $c_1 = 0$, $c_2 = \frac{23}{12}$, $c_3 = -\frac{1}{2}$, $c_4 = 0$ giving

$$g_{tt}^{\star(1)}(r) = -\,\frac{23\,r^2\,+\,11\,r\,+\,6}{12\,r^3}\,, \tag{5.24}$$
$$g_{rr}^{\star(1)}(r) = -\,\frac{r^2\,+\,7\,r\,+\,10}{12\,r^3}\,, \tag{5.25}$$

[8] One star is used upon the metric and dilaton with these boundary conditions. We only review these metrics and dilaton and we are not going to use them.

$$\phi^{\star(1)}(r) = -\frac{2 + 3r + 6r^2}{12r^3}. \qquad (5.26)$$

The boundary conditions imposed in (5.23) produce finite corrections to the Hawking temperature and the entropy of the black-hole. In addition it produces corrections to the newtonian mass which is provided by the asymptotic behaviour of $g_{tt}(r)$ at large r.

In the next section we are going first to calculate the α' corrections to the time-dual of the Schwarzschild metric and then we will use the α' modified T-duality rules. We obtained these rules by studying the α' corrections to the Kasner background. Within the α'-calculations we fixed the asymptotic behaviours at infinity both for the Kasner background and its dual. To be consistent with those calculations and due to the intrinsic singularity of the time-dual of the Schwarzschild metric we use the following boundary conditions on the field contents of the Schwarzschild background (as opposed to the boundary condition in (5.23)),

$$\begin{aligned}
\phi(r)|_{r=\infty} &= 0\,, \\
g_{tt}(r)|_{r\sim\infty} &= 1 - \frac{1}{r} + O\left(\frac{1}{r^2}\right), \\
g_{rr}(r)|_{r\sim\infty} &= 1 + \frac{1}{r} + O\left(\frac{1}{r^2}\right). \qquad (5.27)
\end{aligned}$$

Choosing the above boundary conditions and using the β functions (2.10), (2.11) identifies the linear α' corrections

$$\begin{aligned}
g_{tt}^{(1)}(r) &= \frac{6 + 5r + 12r^2 - 12r^3 + (18r^3 - 12r^4)\ln(1-\frac{1}{r})}{12\,r^3\,(r-1)}\,, \\
g_{rr}^{(1)}(r) &= \frac{10 - 3r - 6r^2 - 6r^3\ln(1-\frac{1}{r})}{12\,r^3\,(r-1)}\,, \\
\phi^{(1)}(r) &= -\frac{1 + \frac{3}{2}r + 3r^2 + 3r^3\ln(1-\frac{1}{r})}{6\,r^3} \qquad (5.28)
\end{aligned}$$

as well as the quadratic α' corrections

$$\begin{aligned}
g_{tt}^{(2)}(r) =\ & \frac{7050 - 5758r + 8125r^2 - 2757r^3 + 1940r^4 + 8140r^5 - 19680r^6 + 6240r^7}{7200\,r^6\,(r-1)^2} + \\
& + \ln\left(1-\frac{1}{r}\right)\frac{120 - 45\,r + 45\,r^2 + 126\,r^3 - 320\,r^4 + 104\,r^5}{120\,r^3\,(r-1)^2} + \\
& + \left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\frac{4\,r - 9}{8\,(r-1)^2}\,, \qquad (5.29) \\
g_{rr}^{(2)}(r) =\ & \frac{6250 - 10154\,r + 5049\,r^2 - 5135\,r^3 + 1880\,r^4 - 2460\,r^5 - 480\,r^6}{7200\,r^6\,(r-1)^2} - \\
& - \ln\left(1-\frac{1}{r}\right)\frac{85 - 75\,r + 22\,r^2 + 8\,r^3}{120\,r^2\,(r-1)^2} + \left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\frac{1+r}{8\,(r-1)^2}\,, \qquad (5.30)
\end{aligned}$$

$$\phi^{(2)}(r) = \frac{-225 - 327\,r - 513\,r^2 - 205\,r^3 - 710\,r^4 - 3930\,r^5 + 4260\,r^6}{3600\,r^6\,(r-1)} + $$
$$+ \ln\left(1 - \frac{1}{r}\right)\,\frac{-5 + 5\,r + 15\,r^2 - 101\,r^3 + 71\,r^4}{60\,r^3\,(r-1)}\,. \tag{5.31}$$

Inserting the above linear and quadratic α' corrections in (5.17), (5.18), (5.20) identifies the quadratic α' corrected Schwarzschild background. The asymptotic behaviours of the fields of the quadratic α' corrected Schwarzschild background at large r follow

$$g_{rr}(r)\,\left(1 - \frac{1}{r}\right) = 1 + \frac{40\,r^2 + 45\,r + 49}{40\,r^6}\,\alpha' + \frac{9}{8\,r^6}\alpha'^2 + O\left(\frac{1}{r^7}\right), \tag{5.32}$$
$$g_{tt}(r)\,\frac{1}{1 - \frac{1}{r}} = 1 + \frac{30\,r^2 + 9\,r - 7}{120\,r^6}\,\alpha' + \frac{3}{8\,r^6}\alpha'^2 + O\left(\frac{1}{r^7}\right), \tag{5.33}$$
$$\phi(r) = \frac{105\,r^3 + 84\,r^2 + 70\,r + 60}{840\,r^7}\,\alpha' + \frac{1}{168\,r^7}\,\alpha'^2 + O\left(\frac{1}{r^8}\right). \tag{5.34}$$

Looking carefully at these expressions one notices that at an α' dependent location outside what was the horizon at $r = 1$ the component g_{tt} of the metric passes through zero. At this point both g_{rr} and $\phi(r)$ remain finite. If the zero in g_{tt} happens at $r = r_0$ then defining a new coordinate $\rho = r - r_0$ the metric near this zero takes the form,

$$ds^2 = -\rho dt^2 + d\rho^2 + c^2 d\omega^2\,, \tag{5.35}$$

where c is a real constant. This metric has a singular Ricci scalar at $\rho = 0$ and thus the α' corrections to the Schwarzschild metric appear to give rise to a naked singularity. The fact that the generic α' corrections to the Schwarzschild metric have singularities outside what was the horizon at $r = 1$ was already noted in [25].

5.2 The linear and the quadratic α' corrections to the time-dual of the Schwarzschild background in $D = 4$

The time dual of the Schwarzschild metric in $D = 4$ is

$$ds^2 = -\frac{1}{1 - \frac{1}{r}}\,dt^2 + \frac{1}{1 - \frac{1}{r}}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\;d\phi^2)\,, \tag{5.36}$$
$$\phi = -\frac{1}{2}\ln\left(1 - \frac{1}{r}\right). \tag{5.37}$$

The Ricci scalar curvatures shows intrinsic singularities at $r = 0$ and at $r = 1$,

$$R = \frac{1}{r^2\,(-r+1)}\,. \tag{5.38}$$

The local string coupling constant in this background reads

$$g_s = e^{+\phi} = \frac{g_0}{\sqrt{1 - \frac{1}{r}}}\,. \tag{5.39}$$

Therefore far away from $r = 1$ both the string theory is perturbative and the space-time is asymptotically flat. Thus within this regime the time-dual of the Schwarzschild metric can be extended to the following perturbative background in the critical bosonic string theory

$$\begin{aligned} ds^2 &= -\,G_{tt}(r)\;dt^2 + G_{rr}(r)\;dr^2 + G_\Omega(r)\;(d\theta^2 + \sin^2\theta\;d\phi^2) + dx_\perp^2\,, \\ \phi &= \Phi(r)\,, \end{aligned}$$

where the metric and the dilaton admit the following perturbative series in α'

$$G_{tt}(r) = \frac{1}{1-\frac{1}{r}}\;\left(1 + \alpha'\;G_{tt}^{(1)}(r) + \;\alpha'^2\;G_{tt}^{(2)}(r) + \cdots\right), \tag{5.40}$$

$$G_{rr}(r) = \frac{1}{1-\frac{1}{r}}\;\left(1 + \alpha'\;G_{rr}^{(1)}(1) + \alpha'^2\;G_{rr}^{(2)}(r) + \cdots\right), \tag{5.41}$$

$$G_\Omega(r) = r^2\,, \tag{5.42}$$

$$\Phi(r) = \;-\frac{1}{2}\ln\left(1 - \frac{1}{r}\right) + \alpha'\;\Phi^{(1)}(r) + \alpha'^2\;\Phi^{(2)}(r) + \cdots\,. \tag{5.43}$$

We fix the asymptotic behaviours of the metric and the dilaton at large r by

$$\Phi(r)|_{r\sim\infty} = \frac{1}{2}\frac{1}{r} \;+ O\left(\frac{1}{r^2}\right), \tag{5.44}$$

$$G_{tt}(r)|_{r\sim\infty} = 1 + \frac{1}{r} \;+ O\left(\frac{1}{r^2}\right), \tag{5.45}$$

$$G_{rr}(r)|_{r\sim\infty} = 1 + \frac{1}{r} \;+ O\left(\frac{1}{r^2}\right). \tag{5.46}$$

In [26] the linear α' corrected Schwarzschild metric computed in [25] with the boundary condition provided in (5.23) and the linear α' modified T-duality is used to obtain the linear α' corrections to the time-dual of the Schwarzschild metric in $D = 4$. This procedure means choosing a specific boundary condition for the time-dual of the Schwarzschild metric at $r = 1$. However we think that due to the intrinsic singularity at $r = 1$ it is not reasonable to set any boundary condition at $r = 1$ in the time-dual of the Schwarzschild background.

Using the β-functions (2.10), (2.11) and the above asymptotic behaviours identifies the linear α' corrections

$$G_{tt}^{(1)}(r) = \frac{\frac{5}{4} + 3\;r - 3\;r^2 - (3\;r^3 - \frac{9}{2}\;r^2)\;\;\ln(1-\frac{1}{r})}{3\;r^2\;(-\;r + 1)}\,, \tag{5.47}$$

$$G_{rr}^{(1)}(r) = \frac{10\;-\;3r\;-\;6r^2\;-6r^3\;\ln(1-\frac{1}{r})}{12\;r^3\;(r\;-\;1)}\,, \tag{5.48}$$

$$\Phi^{(1)}(r) = \frac{1 - 3\;r - 6\;r^2 - 6\;r^3\;\;\ln(1-\frac{1}{r})}{24\;r^3\;(r-1)}\,, \tag{5.49}$$

as well as the quadratic α' corrections

$$
\begin{aligned}
G^{(2)}_{tt}(r) \;=\;& \frac{300-2042\,r+2125\,r^2+1557\,r^3-740\,r^4-22540\,r^5+26880\,r^6-6240\,r^7}{7200\;r^6\;(r-1)^2}\;+\\
&+\;\ln\left(1-\frac{1}{r}\right)\;\frac{75+215r-726r^2+560r^3-104r^4}{120\;r^2\;(r-1)^2}\;+\\
&+\left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\;\frac{9-11r+4r^2}{8\;(r-1)^2}\,,
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
G^{(2)}_{rr}(r) \;=\;& \frac{2200-4754\,r+\;5049\,r^2-5135\,r^3+1880\,r^4-2460\,r^5-480\,r^6}{7200\;r^6\;(r-1)^2}\;+\\
&+\;\ln\left(1-\frac{1}{r}\right)\;\frac{-85+75r-22r^2-8r^3}{120\;r^2\;(r-1)^2}+\left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\;\frac{r+1}{8\;(r-1)^2}\,,
\end{aligned}
\tag{5.51}
$$

$$
\begin{aligned}
\Phi^{(2)}(r) \;=\;& \frac{-1350+2116\,r-456\,r^2-875\,r^3+680\,r^4-2460\,r^5-9480\,r^6+10800\,r^7}{14400\,r^6\;(r-1)^2}\;+\\
&+\ln\left(1-\frac{1}{r}\right)\;\frac{5-10\,r+45\,r^2+38\,r^3-248\,r^4+180\,r^5}{240\,r^3\;(r-1)^2}\;+\\
&+\left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\frac{r}{16(r\;-\;1)^2}\,.
\end{aligned}
\tag{5.52}
$$

5.3 Applying the quadratic α' modified T-duality on the α'^2 corrected Schwarzschild background and its dual in $D = 4$

Earlier we obtained the α' modified T-duality rules for time-dependent geometries. These same rules should also describe T-duality in the Schwarzschild metric, but now the T-duality acts in the direction of the time-like Killing vector outside the black hole horizon. These include the Euclidean geometry of the Schwarzschild metric. Consequently the α' modified T-duality rules can be legitimately applied to the Schwarzschild background. In order to do this we first introduce the analog of the co-moving and the almost co-moving frame for the metrics of the Schwarzschild background and its time-dual.

The quadratic α' corrected Schwarzschild metric and its time-dual are spherically symmetric. A spherically symmetric metric can be written in a coordinate where the radial component of the metric is the identity. This coordinate is the analog of the co-moving frame which we are going to refer to as the "*radial co-moving frame*". On the other hand the "*almost radial co-moving frame*" is defined as a coordinate which can be transformed to the radial co-moving frame by a *single* re-parametrization of the radial direction . In the almost radial co-moving frame the metric reads

$$
ds^2 = f(r)\; dr^2 + ds^2_\perp\,,
\tag{5.53}
$$

where $f(r)$ is called the " characteristic function" of the almost radial co-moving frame. The quadratic α' Schwarzschild metric in (5.15) and its time-dual in (5.40) are already in the almost radial co-moving frame respectively with $g_{rr}(r)$ and $G_{rr}(r)$ as their characteristic functions. These two characteristic functions are not equal:

$$
G_{rr}(r) - g_{rr}(r) = \frac{3\,(4\,r\;-\;3)\;\alpha'^2}{16\;(r\;-\;1)^3\,r^5}\,,
\tag{5.54}
$$

In order to apply the α' modified T-duality rules to the α' corrected Schwarzschild metric and its α' corrected time-dual we should first write them in the almost radial co-moving frame with the same characteristic functions. In the following, we write the quadratic α' corrected time-dual of the Schwarzschild metric in the almost radial co-moving frame with the characteristic function of the quadratic α' corrected Schwarzschild metric.

The α' corrected time-dual of the Schwarzschild metric in the new coordinate provided by

$$r \to r - \frac{\alpha'^2}{16\,(r\,-\,1)\,r^5}\,, \tag{5.55}$$

reads

$$\begin{aligned} ds^2 &= -\,\tilde{g}_{tt}(r)\;dt^2 + g_{rr}(r)\;dr^2 + \tilde{g}_\Omega(r)\;(d\theta^2\,+\,\sin^2\theta\,d\phi^2)\,,\\ \phi &= \tilde{\phi}(r)\,, \end{aligned} \tag{5.56}$$

where $g_{rr}(r)$ is given by (5.18). Other components of the metric and the dilaton read

$$\tilde{g}_{tt}(r) \;=\; \frac{1}{1-\frac{1}{r}}\;\left(1+\alpha'\;\tilde{g}^{(1)}_{tt}(r) +\;\alpha'^2\;\tilde{g}^{(2)}_{tt}(r)\right), \tag{5.57}$$

$$\tilde{g}_\Omega(r) \;=\; r^2 + \frac{\alpha'^2}{8\,r^4\,(r-1)}\,, \tag{5.58}$$

$$\tilde{\phi}(r) \;=\; -\frac{1}{2}\ln\left(1-\frac{1}{r}\right) + \alpha'\;\tilde{\phi}^{(1)}(r) + \alpha'^2\;\tilde{\phi}^{(2)}(r)\,. \tag{5.59}$$

The α' coefficients to the dilaton $\phi(r)$ are

$$\tilde{\phi}^{(1)}(r) \;=\; \frac{1\,-\,3\,r\,-\,6\,r^2\,-\,6\,r^3\,\ln(1-\frac{1}{r})}{24\,r^3\,(r\,-\,1)}\,, \tag{5.60}$$

$$\begin{aligned} \tilde{\phi}^{(2)}(r) \;=\; &\frac{-1800+2116r-456r^2+10800r^7-875r^3+680r^4-2460r^5-9480r^6}{14400\,r^6\,(r-1)^2}+\\ &+\ln\left(1-\frac{1}{r}\right)\;\frac{-248\,r^4+180\,r^5+38\,r^3-10\,r+45\,r^2+5}{240\;r^3\,(r-1)^2}+\\ &+\left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\;\frac{r}{16\,(r-1)^2}\,. \end{aligned}$$

The α' coefficients to the $\tilde{g}_{tt}(r)$ read

$$\tilde{g}^{(1)}_{tt}(r) \;=\; \frac{-\,5\,-\,12\,r\,+\,12\,r^2\,+\,(12\,r^3\,-\,18\,r^2)\,\ln(1\,-\,\frac{1}{r})}{12\,r^2\,(r\,-\,1)}\,, \tag{5.61}$$

$$\begin{aligned} \tilde{g}^{(2)}_{tt}(r) \;=\; &-\frac{150+2042\,r-2125\,r^2-1557\,r^3+740\,r^4+22540\,r^5-26880\,r^6+6240\,r^7}{7200\,r^5\,(r-1)^3}+\\ &+\ln\left(1-\frac{1}{r}\right)\;\frac{215\,r+75-104\,r^4-726\,r^2+560\,r^3}{120\,r\,(r-1)^3}+\\ &+\left\{\ln\left(1-\frac{1}{r}\right)\right\}^2\frac{r\,(-11\,r+9+4\,r^2)}{8\,(r-1)^3}\,. \end{aligned} \tag{5.62}$$

This new coordinate is the almost radial co-moving frame with the characteristic function $g_{rr}(r)$. Now the α' modified T-duality rules provided by (4.24)–(4.26) can be applied to the quadratic α' corrected Schwarzschild background (5.15) and its time-dual (5.56).

The tt, $\phi\phi$ and $\theta\theta$ components of the quadratic α' corrected Schwarzschild background (5.15) and its time dual (5.56) satisfy the T-duality rules given by (4.24) and (4.25)

$$\begin{aligned}\ln \tilde{g}_{tt} - \frac{\alpha'}{4}\, \tilde{\nabla}_\mu \ln \tilde{g}_{tt} \,.\, \tilde{\nabla}^\mu \ln \tilde{g}_{tt} &= \\ &= -\left\{\ln g_{tt} - \frac{\alpha'}{4}\, \nabla_\mu \ln g_{tt} \,.\, \nabla^\mu \ln g_{tt}\right\} + O(\alpha'^2 \nabla^4)\,, \qquad (5.63)\\ \ln \tilde{g}_{ii} + \frac{\alpha'^2}{32} \tilde{\nabla}_\mu \ln \tilde{g}_{tt} . \tilde{\nabla}^\mu \ln \tilde{g}_{ii}\ \tilde{\nabla}_\nu \ln \tilde{g}_{tt} . \tilde{\nabla}^\nu \ln \tilde{g}_{tt} &= \\ &= \ln g_{ii} + \frac{\alpha'^2}{32} \nabla_\mu \ln g_{ii} . \nabla^\mu \ln g_{tt}\ \nabla_\nu \ln g_{tt} . \nabla^\nu \ln g_{tt} + O(\alpha'^3 \nabla^6)\,. \quad (5.64)\end{aligned}$$

Note that in (4.26) we should substitute $\det g$ and $\det \tilde{g}$ respectively with $\det^* g$ and $\det^* \tilde{g}$ given by

$$det^* g = \frac{\det g}{g_{rr}(r)}\,, \qquad (5.65)$$

$$det^* \tilde{g} = \frac{\det \tilde{g}}{g_{rr}(r)}\,. \qquad (5.66)$$

In order to check (4.26) we first calculate

$$\tilde{\phi}(r) - \frac{1}{4} \ln \det \tilde{g} - \phi(r) + \frac{1}{4} \ln \det g = \frac{3 - 4r}{32\, r^6\, (r - 1)^2}\, \alpha'^2\,. \qquad (5.67)$$

Also the α'^2 terms in the l.h.s. of (4.26)

$$\begin{aligned}\tilde{\nabla}_\mu \ln \tilde{g}_{tt}\ \tilde{\nabla}_\nu \ln \tilde{g}_{tt}\ \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{tt} &= \frac{4r - 3}{2\, r^6\, (r - 1)^2} + O(\alpha')\,, \quad (5.68)\\ \tilde{\nabla}_\mu \ln \tilde{g}_{tt} . \tilde{\nabla}^\mu \ln \tilde{g}_{tt}\ \tilde{\nabla}_\nu \ln \tilde{g}_{tt} . \tilde{\nabla}^\nu \left(\tilde{\phi} - \frac{1}{4} \ln \det^* \tilde{g}\right) &= \frac{4r - 5}{4 r^6\, (r - 1)^2} + O(\alpha')\,, \quad (5.69)\\ \tilde{\nabla}_\mu \ln \tilde{g}_{tt} . \tilde{\nabla}^\mu \ln \tilde{g}_{tt}\ \tilde{\Box} \ln \tilde{g}_{tt} &= \frac{1}{r^6\, (r - 1)^2} + O(\alpha')\,, \quad (5.70)\end{aligned}$$

and the α'^2 terms in the r.h.s. of (4.26)

$$\begin{aligned}\nabla_\mu \ln g_{tt}\ \nabla_\nu \ln g_{tt}\ \nabla^\mu \nabla^\nu \ln g_{tt} &= -\frac{4r - 3}{2\, r^6\, (r - 1)^2} + O(\alpha')\,, (5.71)\\ \nabla_\mu \ln g_{tt} . \nabla^\mu \ln g_{tt}\ \nabla_\nu \ln g_{tt} . \nabla^\nu \left(\phi - \frac{1}{4} \ln \det^* g\right) &= -\frac{4r - 3}{4 r^6\, (r - 1)^2} + O(\alpha')\,, (5.72)\\ \nabla_\mu \ln g_{tt} . \nabla^\mu \ln g_{tt}\ \Box \ln g_{tt} &= 0 + O(\alpha')\,. \qquad (5.73)\end{aligned}$$

Inserting the above expressions in (4.26) results the following equations for A, B and C

$$\left.\begin{aligned} A + B + C &= 1\\ A + B &= 1\\ 3A + 4B - C &= 3\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} A &= 1\\ B &= 0\\ C &= 0\end{aligned}\right. \qquad (5.74)$$

which identifies $A = 1$ and $B = C = 0$. Substituting these values in (4.26) results

$$\tilde{\phi} - \frac{1}{4} \ln \det \tilde{g} + \frac{\alpha'^2}{32} \tilde{\nabla}_\mu \ln \tilde{g}_{25\,25} \ \tilde{\nabla}_\nu \ln \tilde{g}_{25\,25} \ \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{25\,25} = \tag{5.75}$$
$$= \phi - \frac{1}{4} \ln \det g + \frac{\alpha'^2}{32} \nabla_\mu \ln g_{25\,25} \ \nabla_\nu \ln g_{25\,25} \ \nabla^\mu \nabla^\nu \ln g_{25\,25} + O(\alpha'^3 \nabla^6)\,.$$

The fact that we can find a consistent assignment of the constants A, B, C and satisfy (5.63) and (5.64) is a nontrivial check on the consistency of our procedure.

As mentioned earlier a general proof of the validity of T-duality at up to third order in α' is presented in [16]. This implies that once we have fully fixed the constants A, B and C, there is no need to check further that this form is consistent with the T-duality for other metrics. One may of course argue that applying T-duality to the Schwarzschild metric is not completely conventional. To resolve this possible ambiguity one could follow the above algorithm applied to another homogeneous cosmology for example of the type studied in [27] where T-duality and cosmology is studied in some detail.

6. Conclusion

In this work the linear and the quadratic α'-corrections to the general diagonal Kasner background in the critical bosonic string theory at the tree level of string interactions are computed.

The asymptotic behaviours of the dilaton and the metric are fixed at *infinity* and the linear and the quadratic α' corrections to the Schwarzschild metric in $D = 4$ are computed. Regardless of the mass of the black hole these corrections diverge at the horizon.

By applying T-duality in the time direction of the Schwarzschild metric the "time-dual of the Schwarzschild metric" is introduced. It is observed that the coordinate singularity of the Schwarzschild metric at the horizon changes into an intrinsic singularity in its T-dual metric.

We find it quite curious that the consistency of α' corrections and T-duality leads us to the conclusion that the α' corrected Schwarzschild metric with singular horizon plays a more important role than that fine tuned to have a smooth horizon. To understand the possible relevance of this result one would need to study string theory on such backgrounds to verify the actual presence of T-duality — equivalence of physics between T-dual geometries under the interchange of momentum and winding modes.

The asymptotic behaviours of the fields of the time-dual of the Schwarzschild metric are fixed at infinity and the linear and the quadratic α' corrections to the time-dual of the Schwarzschild metric in $D = 4$ are computed.

The time-dual of the Schwarzschild metric is massless in $D = 4$. As a generalization of this geometry we introduced the following background

$$ds^2 = -\,dt^2 + \frac{dr^2}{1 + \left(\frac{q}{r^{D-3}}\right)^2} + r^2\, d\Omega_{D-2} + dx^2_{26-D}\,,$$
$$\phi = \pm \frac{D-2}{2\sqrt{D-3}} \,\mathrm{ArcSinh}\left(\frac{q}{r^{D-3}}\right). \tag{6.1}$$

The above background represent a non-trivial (and singular) massless geometry in an arbitrary dimension for the low energy gravitational theory of the bosonic string.

The quadratic α' modification to the rules of T-duality for time dependent backgrounds composed of dilaton and diagonal metric are obtained. If we choose the co-moving frame to represent the dilaton and the metric

$$
\begin{aligned}
ds^2 &= -\,dt^2 + \sum_{i=1}^{25} g_{ii}(t)\, dx_i^2\,, \\
\phi &= \phi(t)\,,
\end{aligned}
\tag{6.2}
$$

and choose the co-moving frame in the T-dual space to represent the T-dual dilaton and metric

$$
\begin{aligned}
d\tilde{s}^2 &= -\,dt^2 + \sum_{i=1}^{25} \tilde{g}_{ii}(t)\, dx_i^2\,, \\
\tilde{\phi} &= \tilde{\phi}(t)\,,
\end{aligned}
\tag{6.3}
$$

then in the string frame the leading α' modification to rules describing T-duality in x_{25} direction and relating (6.2) to (6.3) when the β functions ,(2.10) and (2.11), are calculated by the dimensional regularization method in the minimal subtraction scheme read

$$
\begin{aligned}
&\ln \tilde{g}_{25\,25} - \frac{\alpha'}{4}\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} = \\
&\quad = -\left\{ \ln g_{25\,25} - \frac{\alpha'}{4}\nabla_\mu \ln g_{25\,25}.\nabla^\mu \ln g_{25\,25} \right\} + O(\alpha'^2\nabla^4)\,,
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
&\ln \tilde{g}_{ii} + \frac{\alpha'^2}{32}\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}.\tilde{\nabla}^\mu \ln \tilde{g}_{ii}\ \tilde{\nabla}_\nu \ln \tilde{g}_{25\,25}.\tilde{\nabla}^\nu \ln \tilde{g}_{25\,25} = \\
&\quad = \ln \tilde{g}_{ii} + \frac{\alpha'^2}{32}\nabla_\mu \ln g_{ii}.\nabla^\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}.\nabla^\nu \ln g_{25\,25} + O(\alpha'^3\nabla^6)\,,
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
&\tilde{\phi} - \frac{1}{4}\ln \det \tilde{g} + \frac{\alpha'^2}{32}\ \tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\ \tilde{\nabla}_\nu \ln \tilde{g}_{25\,25}\ \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{25\,25} = \\
&\quad = \phi - \frac{1}{4}\ln \det g + \frac{\alpha'^2}{32}\ \nabla_\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}\ \nabla^\mu \nabla^\nu \ln g_{25\,25} + O(\alpha'^3\nabla^6)\,.
\end{aligned}
\tag{6.6}
$$

As mentioned before within the convention provided by the β functions, (6.4) means that applying T-duality in a general time-dependent background alters the metric in all directions. These rules are consistent with the linear α' corrections to T-duality studied in [3, 4], and have also been checked to agree with T-duality and quadratic α' corrections for the Schwarzschild metric in $D = 4$ and $D = 5$ space-time dimensions.

In case of the Kasner background, keeping the quadratic α' terms in (6.4) results in

$$
\begin{aligned}
&\ln \tilde{g}_{25\,25} - \frac{\alpha'}{4}\,\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} + \frac{\alpha'^2}{32}(\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\tilde{\nabla}^\mu \ln \tilde{g}_{25\,25})^2 - \frac{\alpha'^2\,\tilde{p}_{25}^2}{2\,t^4} = \\
&= -\left\{ \ln g_{25\,25} - \frac{\alpha'}{4}\nabla_\mu \ln g_{25\,25}\nabla^\mu \ln g_{25\,25} + \frac{\alpha'^2}{32}(\nabla_\mu \ln g_{25\,25}\nabla^\mu \ln g_{25\,25})^2 - \frac{\alpha'^2 p_{25}^2}{2t^4} \right\}.
\end{aligned}
\tag{6.7}
$$

There exist nine possibilities to write $\frac{p_{25}^2}{2\,t^4}$ as derivatives of the dilaton and logarithm of the Kasner metric. Using the quadratic α' corrected Schwarzschild metric and its T-dual in $D = 4$ and $D = 5$ fails to single out a unique possibility. In order to write (6.7) unambiguously for a general time-dependent background one should study the quadratic α' corrections to other backgrounds such as the Schwarzschild metric in higher dimensions, the massless black holes provided in (6.1) or the homogeneous cosmologies studied in [27] and their corresponding T-duals.

As a generalization of this work one may consider the four-loop α' corrections in the critical bosonic string theory where the corresponding β-functions are computed in [28] or the four-loop α' corrections in superstring theory [29]–[34].

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